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A new presymplectic geometrical framework for time-dependent Lagrangian systems: the constraint algorithm and the second-order differential equation problem

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Abstract. A gauge-invariant presymplectic setting for time-dependent Lagrangian systems is proposed. In the resulting geometrical framework, the constraint algorithm developed by Gotay and co-workers is extended to the non-autonomous case. Under the assumption of 'admissibility', the second-order differential equation problem is solved in the newer scheme.

1. Introduction

In recent papers [12, 13], a new mathematical setting for a gauge-invariant formulation of Lagrangian and Hamiltonian mechanics has been proposed. The whole theory relies on the introduction of the so-called bundle of *affine scalars*, namely of a principal fibre bundle P over the configuration spacetime $\mathcal{V}_{n+1}$, with structural group $(\Re, +)$.

The bundle P allows one to construct two further principal fibre bundles $\mathcal{L}(\mathcal{V}_{n+1})$ and $\mathcal{L}^c(\mathcal{V}_{n+1})$ over the velocity space $j_1(\mathcal{V}_{n+1})$, called, respectively, the *Lagrangian* and the *co-Lagrangian* bundle. Denoting by $j_1(P, \Re)$ the first jet-space associated with the fibration $P \to \mathcal{V}_{n+1} \to \Re$, the resulting geometrical set-up is summarized into the commutative diagram

$$\begin{array}{ccc} j_1(P,\Re) & \longrightarrow & \mathcal{L}^c(\mathcal{V}_{n+1}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{L}(\mathcal{V}_{n+1}) & \longrightarrow & j_1(\mathcal{V}_{n+1}) \end{array} \tag{1.1}$$

in which all arrows indicate principal fibrations with structural groups isomorphic to $(\Re, +)$.

The diagram (1.1) provides the starting point for a revisitation of classical Lagrangian mechanics, mainly in connection with gauge-theoretical aspects, resulting in a new geometrical interpretation of such concepts as the Lagrangian function, Poincaré–Cartan 1- and 2-form [12]. More specifically, a Lagrangian L is viewed as the representation of a given *Lagrangian section* l: $j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$, while the associated Poincaré–Cartan 1-form $\theta_l := (L - \frac{\partial L}{\partial \dot q^i}\dot q^i)\,\mathrm{d}t + \frac{\partial L}{\partial \dot q^i}\,\mathrm{d}q^i$ is seen to represent a connection in the principal fibre bundle $\mathcal{L}^c(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$. The curvature of this connection is recognized as a gauge-invariant field on $j_1(\mathcal{V}_{n+1})$, identical (up to a sign) to the Poincaré–Cartan 2-form $\Omega_l := \mathrm{d}\theta_l$.

The construction of the Lagrangian bundles and the algorithm assigning to every Lagrangian section l the corresponding Poincaré–Cartan 1-form are briefly reviewed in section 2.

A first result of this paper is the construction of a gauge-invariant presymplectic formalism for time-dependent Lagrangian mechanics.

To this end, in section 3, we shall focus our attention on the Lagrangian bundle $\mathcal{L}(\mathcal{V}_{n+1})$. In more detail, every Lagrangian section l is seen to induce a corresponding connection on the principal fibre bundle $j_1(P, \Re) \to \mathcal{L}(\mathcal{V}_{n+1})$. Under the usual regularity condition, up to a sign, the curvature of such a connection endows $\mathcal{L}(\mathcal{V}_{n+1})$ with a symplectic structure $\tilde{\Omega}_l$. When the regularity hypothesis is violated (i.e. when the Lagrangian section is 'degenerate'), consistently with what is commonly done in the literature, we shall assume that the 2-form $\tilde{\Omega}_l$ is presymplectic.

By means of the 2-form $\tilde{\Omega}_l$ we may set up a pseudo-'problem of motion' on the bundle $\mathcal{L}(\mathcal{V}_{n+1})$ expressed by the equation

$$\tilde{Z} \rfloor \tilde{\Omega}_l = -\mathrm{d}\varphi_l \tag{1.2}$$

where $\tilde{Z} \in D^1(\mathcal{L}(\mathcal{V}_{n+1}))$ is the unknown and φ_l (identified with the trivialization of the principal fibre bundle $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ induced by the section l) plays the role of a 'Hamiltonian' on $\mathcal{L}(\mathcal{V}_{n+1})$.

The mathematical equivalence between the problem described above and the standard one formulated in the velocity space $j_1(\mathcal{V}_{n+1})$ by means of the cosymplectic (precosymplectic) structure $(\Omega_l, \mathrm{d}t)$ [4, 5, 10, 12, 18, 19] is proved both in the regular and in the singular case.

We note that the algorithm generating equation (1.2) is the Lagrangian counterpart of the one proposed in [13] for time-dependent Hamiltonian mechanics.

When $\tilde{\Omega}_l$ is presymplectic, equation (1.2) may or may not admit solutions. Moreover, in general, even in the former case, the solution will not be unique.

To cope with this fact, in section 3.2 we have extended to the Lagrangian bundle the constraint algorithm developed by Gotay, *et al* in [1]. The idea is to work on the surface $M_0 := l(j_1(\mathcal{V}_{n+1})) \subset \mathcal{L}(\mathcal{V}_{n+1})$, image of $j_1(\mathcal{V}_{n+1})$ under the given Lagrangian section l, and use the presymplectic structure to find whether there exists a submanifold $M \subset \mathcal{L}(\mathcal{V}_{n+1})$ along which equation (1.2) holds and—in the affirmative case—to characterize it in convenient geometrical terms.

For different generalizations of the constraint algorithm to the non-autonomous case—based essentially on the precosymplectic geometry—see [4, 5] and references therein.

As is well known, finding a 'final constraint manifold' M along which equation (1.2) admits solutions is not enough. Indeed, in principle, such solutions may have no dynamical meaning at all.

The search for kinematically admissible solutions of equation (1.2), referred to in the literature as 'the second-order differential equation (SODE) problem', is dealt with in section 3.3 under the assumption of *admissibility* of the Lagrangian section l in the sense of [3]. Borrowing from [3] and adapting the argument to the present geometrical context, we prove the existence of at least one submanifold S of M along which equation (1.2) possesses a unique SODE solution.

In this connection, we recall that, in the non-autonomous case, a solution of the second-order differential equation problem has been presented in [4, 5]. The difference with the present analysis is that in [4] the authors examine time-dependent Lagrangians defined on trivial fibred manifolds $T(Q) \times \Re$, while in [5] they work under the assumption of *almost regularity*, which is slightly more restrictive than that of admissibility.

2. Geometrical preliminaries

2.1. The Lagrangian bundles

We present here a brief review of the construction of the so-called *Lagrangian bundles*, which, as we shall see, provide a natural mathematical framework for a gauge-invariant presymplectic formulation of time-dependent Lagrangian systems. For a more detailed account of these bundles, the reader is referred to [12].

To start with, let $\mathcal{V}_{n+1}$ be the configuration spacetime of a holonomic system $\mathcal{B}$ with n degrees of freedom. As is well known [5, 9–16], the $(n+1)$-dimensional manifold $\mathcal{V}_{n+1}$ carries a natural fibration $t : \mathcal{V}_{n+1} \to \Re$ over the real line, identified with the absolute time function.

We associate with $\mathcal{B}$ a principal fibre bundle $\pi : P \to \mathcal{V}_{n+1}$, called the bundle of *affine scalars* over $\mathcal{V}_{n+1}$, with structural group $(\Re, +)$, diffeomorphic (in a non-canonical way) to the Cartesian product $\mathcal{V}_{n+1} \times \Re$.

Using the additive notation

$$(\xi, \nu) \in \Re \times P \to \psi_\xi(\nu) := \nu + \xi \in P \tag{2.1}$$

to denote the group action of $(\Re, +)$ on P, a function $u : P \to \Re$ is called a trivialization of P if and only if it satisfies the requirement

$$u(\nu + \xi) = u(\nu) + \xi. \tag{2.2}$$

Given any trivialization u of P, we may lift every local coordinate system $t, q^1, \ldots, q^n$ defined on an open subset $U \subset \mathcal{V}_{n+1}$ to a *fibred coordinate system* $t, q^1, \ldots, q^n, u$ on $\pi^{-1}(U) \subset P$. The most general fibred coordinate transformation is then described by the relations

$$\bar{t} = t + c \qquad \bar{q}^i = \bar{q}^i(t, q^1, \ldots, q^n) \qquad \bar{u} = u + f(t, q^1, \ldots, q^n) \tag{2.3}$$

with $f(t, q^i) \in F(\mathcal{V}_{n+1})$. At the same time, in fibred coordinates, the group action (2.1) is expressed as

$$t(\nu + \xi) = t(\nu) \qquad q^i(\nu + \xi) = q^i(\nu) \qquad u(\nu + \xi) = u(\nu) + \xi. \tag{2.4}$$

By pull-back, the bundle P inherits from $\mathcal{V}_{n+1}$ the fibration $t : P \to \Re$ over the real line. We denote by $\pi : j_1(P, \Re) \to P$ the first jet space associated with this fibration. As usual, we refer $j_1(P, \Re)$ to local jet-coordinates $t, q^i, u, \dot{q}^i, \dot{u}$, subject to the transformation laws

$$\bar{t} = t + c \qquad \bar{q}^i = \bar{q}^i(t, q) \qquad \bar{u} = u + f(t, q) \tag{2.5a}$$

$$\dot{\bar{q}}^i = \frac{\partial \bar{q}^i}{\partial q^k}\dot{q}^k + \frac{\partial \bar{q}^i}{\partial t} \qquad \dot{\bar{u}} = \dot{u} + \frac{\mathrm{d}f}{\mathrm{d}t} \tag{2.5b}$$

with $\frac{\mathrm{d}f}{\mathrm{d}t} := \frac{\partial f}{\partial q^k}\dot{q}^k + \frac{\partial f}{\partial t}$.

The geometrical properties of the manifold $j_1(P, \Re)$ include, in the first place, all attributes related to the jet-structure [10, 17]. In particular, for later use, we recall:

- the contact bundle $C(j_1(P, \Re))$, spanned locally by the 1-forms

$$\omega^0 = \mathrm{d}u - \dot{u}\,\mathrm{d}t \qquad \omega^i = \mathrm{d}q^i - \dot{q}^i\,\mathrm{d}t \qquad i = 1, \ldots, n$$

- the fundamental tensor field $\hat{J}$, expressed locally as

$$\hat{J} = \omega^0 \otimes \frac{\partial}{\partial \dot{u}} + \omega^i \otimes \frac{\partial}{\partial \dot{q}^i}$$

- the fibre differential d_v, acting on an arbitrary function $f \in F(j_1(P, \Re))$ as

$$d_v f = \frac{\partial f}{\partial \dot{u}}\, \omega^0 + \frac{\partial f}{\partial \dot{q}^i}\, \omega^i. \tag{2.6}$$

In addition to this, the principal bundle structure of P, together with the canonical identification

$$j_1(P, \Re) \simeq \{\, X \in T(P) \mid \langle X, \mathrm{d}t \rangle = 1 \,\} \tag{2.7}$$

gives rise to two distinguished actions of the group $\Re$ on $j_1(P, \Re)$.

The first one, denoted by $\psi_{\xi *} : j_1(P, \Re) \to j_1(P, \Re)$ $\xi \in \Re$, is simply the push-forward of the action (2.1) restricted to $j_1(P, \Re) \subset T(P)$. In local coordinates, expressing explicitly the identification (2.7) in the form

$$z = \left[\frac{\partial}{\partial t} + \dot{q}^i(z)\, \frac{\partial}{\partial q^i} + \dot{u}(z)\, \frac{\partial}{\partial u} \right]_{\pi(z)} \in j_1(P, \Re) \tag{2.8}$$

and recalling equation (2.4), we have the representation

$$\psi_{\xi *}(z) = \left[\frac{\partial}{\partial t} + \dot{q}^i(z)\, \frac{\partial}{\partial q^i} + \dot{u}(z)\, \frac{\partial}{\partial u} \right]_{\pi(z)+\xi} \tag{2.9a}$$

summarized into the symbolic relation

$$\psi_{\xi *} : (t, q^i, u, \dot{q}^i, \dot{u}) \to (t, q^i, u + \xi, \dot{q}^i, \dot{u}). \tag{2.9b}$$

The second action of $\Re$ on $j_1(P, \Re)$ is induced by the vector field $\frac{\partial}{\partial u}$, usually referred to as the *fundamental vector field* of P. In view of the affine nature of the fibration $\pi : j_1(P, \Re) \to P$, the latter gives rise to a one-parameter group of diffeomorphisms $\varphi_\xi : j_1(P, \Re) \to j_1(P, \Re)$, based on the relation

$$\varphi_\xi(z) := z + \xi \left(\frac{\partial}{\partial u} \right)_{\pi(z)} \qquad \forall z \in j_1(P, \Re) \quad \xi \in \Re. \tag{2.10}$$

Again, from equation (2.8) we obtain the representation

$$\varphi_\xi(z) = \left[\frac{\partial}{\partial t} + \dot{q}^i(z)\, \frac{\partial}{\partial q^i} + (\dot{u}(z) + \xi)\, \frac{\partial}{\partial u} \right]_{\pi(z)} \tag{2.11a}$$

written more synthetically as

$$\varphi_\xi : (t, q^i, u, \dot{q}^i, \dot{u}) \to (t, q^i, u, \dot{q}^i, \dot{u} + \xi). \tag{2.11b}$$

Denoting by $\mathcal{L}(\mathcal{V}_{n+1})$ and $\mathcal{L}^c(\mathcal{V}_{n+1})$, respectively, the quotient spaces of $j_1(P, \Re)$ with respect to the group actions (2.9) and (2.11), we have the following properties [12]:

- $\mathcal{L}(\mathcal{V}_{n+1})$ and $\mathcal{L}^c(\mathcal{V}_{n+1})$ are fibre bundles over $\mathcal{V}_{n+1}$ with projections $\mathcal{L}(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$ and $\mathcal{L}^c(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$ induced by the composite map $j_1(P, \Re) \to P \to \mathcal{V}_{n+1}$;
- every jet-coordinate system $t, q^i, u, \dot{q}^i, \dot{u}$ on $j_1(P, \Re)$ induces corresponding local coordinates $t, q^i, \dot{q}^i, \dot{u}$ on $\mathcal{L}(\mathcal{V}_{n+1})$ and $t, q^i, u, \dot{q}^i$ on $\mathcal{L}^c(\mathcal{V}_{n+1})$;
- both actions (2.9) and (2.11) make $j_1(P, \Re)$ into a principal fibre bundle (respectively, over $\mathcal{L}(\mathcal{V}_{n+1})$ and over $\mathcal{L}^c(\mathcal{V}_{n+1})$) with structural groups both isomorphic to $(\Re, +)$.

Finally, it is easily seen that the group actions (2.9) and (2.11) commute. Accordingly, the action (2.9) induces a corresponding action on the quotient space $\mathcal{L}^c(\mathcal{V}_{n+1})$, while the action (2.11) induces an action on $\mathcal{L}(\mathcal{V}_{n+1})$. The quotient of $\mathcal{L}^c(\mathcal{V}_{n+1})$ with respect to the action (2.9) and the quotient of $\mathcal{L}(\mathcal{V}_{n+1})$ with respect to the action (2.11) are then the same manifold, clearly identified with the velocity space $j_1(\mathcal{V}_{n+1})$. In this connection, we recall the following [12].

Theorem 2.1. *Both spaces $\mathcal{L}(\mathcal{V}_{n+1})$ and $\mathcal{L}^c(\mathcal{V}_{n+1})$ are principal fibre bundles over the space $j_1(\mathcal{V}_{n+1})$, with structural groups isomorphic to $(\Re, +)$.*

The content of the previous discussion is summarized into the commutative diagram

$$\begin{array}{ccc} j_1(P,\Re) & \longrightarrow & \mathcal{L}^c(\mathcal{V}_{n+1}) \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{L}(\mathcal{V}_{n+1}) & \longrightarrow & j_1(\mathcal{V}_{n+1}) \end{array} \tag{2.12}$$

in which all arrows denote principal fibrations, with structural groups isomorphic to $(\Re, +)$.

Definition 2.1. *The principal fibre bundles $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ and $\mathcal{L}^c(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ are called, respectively, the Lagrangian and the co-Lagrangian bundle over $j_1(\mathcal{V}_{n+1})$*

2.2. The Poincaré–Cartan 1-form

The geometrical environment based on the diagram (2.12) provides the mathematical setting for a formulation of Lagrangian mechanics, automatically embodying the gauge-theoretical aspects of the theory.

Referring once again to [12] for a more detailed discussion, in this subsection we shall outline a brief description of the construction of the so-called Poincaré–Cartan 1-form associated with any given Lagrangian.

The basic idea is to replace the concept of the Lagrangian function $L(t, q^i, \dot{q}^i) \in F(j_1(\mathcal{V}_{n+1}))$ with a *Lagrangian section* $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$, i.e. with a section of the Lagrangian bundle, expressed locally as

$$\dot{u} = L(t, q^i, \dot{q}^i). \tag{2.13}$$

Taking equations (2.5) into account, it is immediately seen that for every change of trivialization $u \to \bar{u} = u + f(t, q^i)$ of the bundle P, the representation (2.13) undergoes the transformation

$$\dot{\bar{u}} = \dot{u} + \frac{\mathrm{d}f}{\mathrm{d}t} = L(t, q^i, \dot{q}^i) + \frac{\mathrm{d}f}{\mathrm{d}t} := L'(t, q^i, \dot{q}^i) \tag{2.14}$$

involving a different, gauge-equivalent Lagrangian L'.

In this way, gauge-equivalent Lagrangians are viewed as different representations—corresponding to different choices of trivialization of P—of the *same* section l†.

Now, given a Lagrangian section l, described locally by equation (2.13), let us consider the trivialization $\varphi_l := \dot{u} - L(t, q^i, \dot{q}^i)$ of the bundle $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ induced by l itself. We may pull-back φ_l on $j_1(P, \Re)$, obtaining the function $\hat{\varphi}_l = \dot{u} - L(t, q^i, \dot{q}^i)$, having the nature of a trivialization of the bundle $j_1(P, \Re) \to \mathcal{L}^c(\mathcal{V}_{n+1})$. Strictly associated with $\hat{\varphi}_l$ is therefore a section $\hat{l} : \mathcal{L}^c(\mathcal{V}_{n+1}) \to j_1(P, \Re)$, still expressed locally as $\dot{u} = L(t, q^i, \dot{q}^i)$.

We have thus seen that every Lagrangian section l may be lifted to a section $\hat{l}$, according to the commutative diagram

$$\begin{array}{ccc} \mathcal{L}^c(\mathcal{V}_{n+1}) & \xrightarrow{\hat{l}} & j_1(P,\Re) \\ \Big\downarrow & & \Big\downarrow \\ j_1(\mathcal{V}_{n+1}) & \xrightarrow{l} & \mathcal{L}(\mathcal{V}_{n+1}). \end{array} \tag{2.15}$$

† This, of course, is a 'passive' way of looking at gauge transformations. For a detailed analysis of the 'active' counterpart, the reader is referred to [12].

Taking equation (2.6) into account, let us now consider the fibre differential of $\hat{\varphi}_l$, expressed in coordinates as

$$\mathrm{d}_v\hat{\varphi}_l = \mathrm{d}_v(\dot{u} - L(t, q^i, \dot{q}^i)) = \omega^0 - \frac{\partial L}{\partial \dot{q}^k}\,\omega^k. \tag{2.16}$$

It is an easy matter to verify that the 1-form (2.16) defines a connection on the principal fibre bundle $j_1(P, \Re) \rightarrow \mathcal{L}(\mathcal{V}_{n+1})$. In fact, it is invariant under the action of the structural group (i.e. under the one-parameter group of diffeomorphisms generated by the field $\frac{\partial}{\partial u}$), and satisfies the duality relation $\langle \mathrm{d}_v\,\hat{\varphi}_l, \frac{\partial}{\partial u}\rangle = 1$.

Making use of the section $\hat{l}$ appearing in the diagram (2.15), we next pull-back the connection 1-form (2.16) on $\mathcal{L}^c(\mathcal{V}_{n+1})$. The final result is then the 1-form

$$\hat{\theta}_l := \hat{l}^*(\mathrm{d}_v\,\hat{\varphi}_l) = \mathrm{d}u - L\,\mathrm{d}t - \frac{\partial L}{\partial \dot{q}^k}\,\omega^k \tag{2.17}$$

defining a connection on the principal fibre bundle $\mathcal{L}^c(\mathcal{V}_{n+1}) \rightarrow j_1(\mathcal{V}_{n+1})$.

To conclude, from equation (2.17), or, more in general, from the theory of connections, the difference $\mathrm{d}u - \hat{\theta}_l$ is easily seen to be the pull-back of a 1-form θ_l on $j_1(\mathcal{V}_{n+1})$, expressed in local coordinates as

$$\theta_l = L\,\mathrm{d}t + \frac{\partial L}{\partial \dot{q}^k}\,\omega^k. \tag{2.18}$$

The latter is immediately recognized as the familiar *Poincaré–Cartan* 1-form associated with the 'Lagrangian' L, involved in the representation (2.13).

If we accomplish an arbitrary change of trivialization, the requirement

$$\hat{\theta}_l = \mathrm{d}\bar{u} - \pi^*(\bar{\theta}_l) = \mathrm{d}(u + f) - \pi^*(\bar{\theta}_l) = \mathrm{d}u - \pi^*(\theta_l) \tag{2.19}$$

yields the transformation law

$$\bar{\theta}_l = \theta_l + \mathrm{d}f \tag{2.20}$$

showing that the really dynamically relevant object over $j_1(\mathcal{V}_{n+1})$ is the *gauge-invariant* Poincaré–Cartan 2-form

$$\Omega_l := \mathrm{d}\theta_l \tag{2.21}$$

identical (up to a sign) to the curvature of the connection (2.17)†.

We shall say that the Lagrangian section $l : j_1(\mathcal{V}_{n+1}) \rightarrow \mathcal{L}(\mathcal{V}_{n+1})$ is *regular* if and only if the Hessian matrix $\left\| \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right\|$ is non-singular.

As is well known (see, for example, [18]), under this assumption the requirements

$$Z \,\lrcorner\, \Omega_l = 0 \qquad \langle Z, \mathrm{d}t\rangle = 1 \tag{2.22}$$

characterize the unique dynamical flow $Z = \frac{\partial}{\partial t} + \dot{q}^i\,\frac{\partial}{\partial q^i} + Z^i\,\frac{\partial}{\partial \dot{q}^i}$ on $j_1(\mathcal{V}_{n+1})$, the solution of the Lagrange's equations

$$Z\left(\frac{\partial L}{\partial \dot{q}^k}\right) - \frac{\partial L}{\partial q^k} = 0.$$

† One comes to the same conclusion by considering 'active' gauge transformations [12].

Remark 2.1. An affine fibration over the spacetime, analogous to the bundle of affine scalars, is used in [20] to set up a suitable geometrical framework for the study of charged particles dynamics. The construction relies entirely on the affine nature of the fibration. In contrast, in the present formulation, a major role is played by a principal fibration. As has been shown, the latter allows one to give a geometrical gauge-invariant description of classical Lagrangian mechanics in terms of sections and connections of principal fibre bundles†. As compared with [20], further advantages of the present approach arise in connection with time-dependent Hamiltonian mechanics, as outlined in [13]. There, the curvature 2-form of a 'canonical' principal connection on the first jet-bundle $j_1(P, \mathcal{V}_{n+1})$ of the fibration $P \to \mathcal{V}_{n+1}$ (the contact fibration in [20]) is proved to endow a suitable 'Hamiltonian' bundle (the phase fibration) with a symplectic structure, providing the basic tool for the implementation of the Hamiltonian counterpart of the theory discussed above.

3. Presymplectic Lagrangian systems

3.1. Equations of motion on $\mathcal{L}(\mathcal{V}_{n+1})$

In this subsection, making use of a presymplectic formalism, we shall construct equations of motion directly on the Lagrangian bundle $\mathcal{L}(\mathcal{V}_{n+1})$.

We shall also discuss the relationships between the resulting scheme and the standard one, formulated on $j_1(\mathcal{V}_{n+1})$ through equations (2.22) [5, 10, 12, 18, 19].

As a preliminary step in the discussion, it is worth outlining some aspects of the geometry of $\mathcal{L}(\mathcal{V}_{n+1})$, which are particularly relevant in the subsequent discussion.

In the first place, we recall that $\mathcal{L}(\mathcal{V}_{n+1})$ is fibred over the configuration spacetime $\mathcal{V}_{n+1}$. The vertical bundle associated with this fibration—henceforth denoted by $V(\mathcal{L}(\mathcal{V}_{n+1}))$—is spanned locally by the vector fields $\frac{\partial}{\partial \dot{u}}, \frac{\partial}{\partial \dot{q}^i}, i = 1, \ldots, n$.

In addition to this, the contact bundle $C(j_1(\mathcal{V}_{n+1}))$ over $j_1(\mathcal{V}_{n+1})$ [10–12], generated locally by the 1-forms $\omega^i = \mathrm{d}q^i - \dot{q}^i\,\mathrm{d}t, i = 1, \ldots, n$, may be pulled-back to $\mathcal{L}(\mathcal{V}_{n+1})$ through the projection $\pi : \mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$. In what follows, for simplicity, we shall preserve the notation $\omega^i = \mathrm{d}q^i - \dot{q}^i\,\mathrm{d}t, i = 1, \ldots, n$, for the pull-back of the 1-forms ω^i on $j_1(\mathcal{V}_{n+1})$.

Further geometrical objects on $\mathcal{L}(\mathcal{V}_{n+1})$ come from the assignment of a Lagrangian section $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$, expressed locally as

$$\dot{u} = L(t, q^i, \dot{q}^i). \tag{3.1}$$

To see this point, denoting by $\varphi_l = \dot{u} - L$ the trivialization of $\mathcal{L}(\mathcal{V}_{n+1})$ determined by the section (3.1), we observe that the differential

$$\mathrm{d}\varphi_l = \mathrm{d}\dot{u} - \mathrm{d}L \tag{3.2}$$

has the nature of a smooth-connection 1-form over the principal fibre bundle $\pi : \mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$. The related horizontal lift associates to every vector field $X = X^0\frac{\partial}{\partial t} + X^i\frac{\partial}{\partial q^i} + \dot{X}^i\frac{\partial}{\partial \dot{q}^i} \in D^1(j_1(\mathcal{V}_{n+1}))$ a corresponding vector field X_l on $\mathcal{L}(\mathcal{V}_{n+1})$, invariant under the action of the structural group (i.e. under the one-parameter group of diffeomorphisms generated by $\frac{\partial}{\partial \dot{u}}$) and expressed locally as

$$X_l = X^0\frac{\partial}{\partial t} + X^i\frac{\partial}{\partial q^i} + \dot{X}^i\frac{\partial}{\partial \dot{q}^i} + X(L)\frac{\partial}{\partial \dot{u}}. \tag{3.3}$$

† Similar ideas to those of the Lagrangian bundle and Lagrangian section may be found in [21] where the Lagrangians are sections of the trivial line bundle $j_1(\mathcal{V}_{n+1}) \times \Re \to j_1(\mathcal{V}_{n+1})$. However, in [21], no gauge considerations are made.

Now, let us return to the connection 1-form

$$\mathrm{d}_v\varphi_l = \omega^0 - \frac{\partial L}{\partial \dot{q}^i}\,\omega^i \tag{3.4}$$

for the principal fibre bundle $\pi : j_1(P, \Re) \to \mathcal{L}(\mathcal{V}_{n+1})$, generated by l through the algorithm discussed in the previous section.

The curvature of the connection (3.4), defined, up to a sign, by the 2-form

$$\tilde{\Omega}_l := -\mathrm{d}(\mathrm{d}_v\varphi_l) = \mathrm{d}\dot{u} \wedge \mathrm{d}t + \mathrm{d}\left(\frac{\partial L}{\partial \dot{q}^i}\right) \wedge \omega^i - \frac{\partial L}{\partial \dot{q}^i}\,\mathrm{d}\dot{q}^i \wedge \mathrm{d}t \tag{3.5}$$

has the nature of an exact 2-form on $\mathcal{L}(\mathcal{V}_{n+1})$, invariant under (passive [12]) gauge transformations $\dot{u} \to \bar{\dot{u}} = \dot{u} + \frac{\mathrm{d}f}{\mathrm{d}t}$. It is a straightforward matter to verify that, when the regularity condition rank $\left\| \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right\| = n$ is satisfied, the 2-form (3.5) has maximal rank, thus endowing $\mathcal{L}(\mathcal{V}_{n+1})$ with a symplectic structure. When this is the case, as pointed out in section 2.2, the section l is said to be a *regular* (or *non-degenerate*) Lagrangian section.

More generally, when the regularity hypothesis is violated, but $\tilde{\Omega}_l$ has *constant rank* everywhere, the 2-form (3.5) is *presymplectic*. In what follows, we shall examine the consequences of this assumption.

Following the standard terminology [2–6], we shall call such an l a *degenerate* (or *singular*) Lagrangian section.

To every (regular or degenerate) Lagrangian section we associate a tensor field of type (1, 1), according to the following construction.

Take any vector field X over $\mathcal{L}(\mathcal{V}_{n+1})$ and lift it horizontally with respect to the connection (3.4), thus obtaining an invariant vector field on $j_1(P, \Re)$ of the form

$$\hat{X} = X + \left(\dot{u}\langle X, \mathrm{d}t\rangle + \frac{\partial L}{\partial \dot{q}^i}\langle X, \omega^i\rangle\right)\frac{\partial}{\partial u}. \tag{3.6}$$

Next, evaluate the image $\hat{J}(\hat{X})$ under the fundamental tensor $\hat{J} = \omega^0 \otimes \frac{\partial}{\partial \dot{u}} + \omega^i \otimes \frac{\partial}{\partial \dot{q}^i}$ of $j_1(P, \Re)$. Finally, project $\hat{J}(\hat{X})$ back to $\mathcal{L}(\mathcal{V}_{n+1})$ by means of the tangent map $T\pi : T(j_1(P, \Re)) \to T(\mathcal{L}(\mathcal{V}_{n+1}))$. The final result is then the vertical vector field

$$T\pi(\hat{J}(\hat{X})) = \langle X, \omega^i\rangle \left(\frac{\partial L}{\partial \dot{q}^i}\,\frac{\partial}{\partial \dot{u}} + \frac{\partial}{\partial \dot{q}^i}\right) \tag{3.7}$$

on $\mathcal{L}(\mathcal{V}_{n+1})$. The correspondence (3.7) depends F-linearly on X, so that, by the quotient law, it defines a tensor field $\tilde{J}$ on $\mathcal{L}(\mathcal{V}_{n+1})$, with local expression

$$\tilde{J} = \omega^i \otimes \left(\frac{\partial L}{\partial \dot{q}^i}\,\frac{\partial}{\partial \dot{u}} + \frac{\partial}{\partial \dot{q}^i}\right). \tag{3.8}$$

On account of equation (3.8) one can easily derive the relations $\tilde{J}\left(\frac{\partial}{\partial \dot{u}}\right) = 0$, $L_{\partial/\partial \dot{u}}\tilde{J} = 0$, indicating that the tensor field $\tilde{J}$ 'projects' onto $j_1(\mathcal{V}_{n+1})$ [6].

More specifically, denoting by $J = \omega^i \otimes \frac{\partial}{\partial \dot{q}^i}$ the fundamental tensor of $j_1(\mathcal{V}_{n+1})$ [5, 10–12, 14, 17, 19], it is immediate to see that $\tilde{J}$ is π-related to J, namely for every $z \in \mathcal{L}(\mathcal{V}_{n+1})$ and every $X \in T_z(\mathcal{L}(\mathcal{V}_{n+1}))$ one has $\pi_*(\tilde{J}_z(X)) = J_{\pi(z)}(\pi_*(X))$. Conversely, it is also true that $(J_z(X))_l = \tilde{J}_{\pi^{-1}(z)}(X_l)$ for every $z \in j_1(\mathcal{V}_{n+1})$ and every $X \in T_z(j_1(\mathcal{V}_{n+1}))$.

After these preliminary remarks, let us now come to the construction of suitable 'equations of motion' on the Lagrangian bundle $\mathcal{L}(\mathcal{V}_{n+1})$.

To this end, given a Lagrangian section l described locally by $\dot{u} = L(t, q^i, \dot{q}^i)$, we look for vector fields $\tilde{Z} \in D^1(\mathcal{L}(\mathcal{V}_{n+1}))$ satisfying the requirement

$$\tilde{Z} \rfloor \tilde{\Omega}_l = -\mathrm{d}\varphi_l. \tag{3.9}$$

Indeed, when l is a *regular* Lagrangian, a straightforward evaluation (left to the reader) shows that equation (3.9) admits the *unique* solution

$$\tilde{Z} = Z + Z(L)\,\frac{\partial}{\partial \dot{u}} \tag{3.10}$$

$Z := \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + Z^i \frac{\partial}{\partial \dot{q}^i}$ denoting the dynamical flow determined by equations (2.22) on $j_1(\mathcal{V}_{n+1})$. In this case, therefore, the vector field $\tilde{Z}$ is uniquely characterized as the horizontal lift (in the sense of equation (3.3)) of the dynamical flow $Z \in D^1(j_1(\mathcal{V}_{n+1}))$.

We thus come to the conclusion that, in the regular case, the problem of motion formulated on $j_1(\mathcal{V}_{n+1})$ through (2.22) and that on $\mathcal{L}(\mathcal{V}_{n+1})$ based on equation (3.9) are equivalent.

More specifically, if $\tilde{Z}$ satisfies equation (3.9) then it is π-projectable on $j_1(\mathcal{V}_{n+1})$, and its image $Z = \pi_*(\tilde{Z})$ is a solution of equations (2.22). Conversely, if Z is a solution of (2.22), its horizontal lift (3.10) satisfies equation (3.9).

The question is now: what happens when the Lagrangian section l is degenerate? In this case, in general, equation (3.9) may admit no solution at all, or, when a solution exists, it may be non-unique.

To account for this situation, in the next subsection we shall set up a presymplectic constraint algorithm, generalizing the one proposed by Gotay and co-workers [1, 2] for autonomous Lagrangian systems.

Prior to this, however, we discuss the relationships between the problems of motion formulated, respectively, on $\mathcal{L}(\mathcal{V}_{n+1})$ and on $j_1(\mathcal{V}_{n+1})$, when the Lagrangian section l is degenerate.

To this end, let us suppose that there exists a maximal submanifold $M \subset \mathcal{L}(\mathcal{V}_{n+1})$ on which equation (3.9) admits a solution X, namely $\forall z \in M\ \exists X \in T_z(\mathcal{L}(\mathcal{V}_{n+1}))$ such that $X \rfloor \tilde{\Omega}_{l|z} = -\mathrm{d}\varphi_{l|z}$

Due to the (straightforward) fact that the forms $\tilde{\Omega}_l$ and $\mathrm{d}\varphi_l$ are invariant under the action of the structural group, i.e. under translations along the fibres of $\mathcal{L}(\mathcal{V}_{n+1})$, one has the following:

Proposition 3.1. *Let $\pi : \mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ denote the canonical projection. Then, for each $z \in M$, the whole fibre $\pi^{-1}(\pi(z))$ over $\pi(z)$ is contained in M.*

Proof. Let $X \in T_z(\mathcal{L}(\mathcal{V}_{n+1}))$ be a solution of equation (3.9). For each $\bar{z} \in \pi^{-1}(\pi(z))$, denote by $\psi_\xi : \mathcal{L}(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ the translation satisfying $\psi_\xi(\bar{z}) = z$, and consider $Y \in T_{\bar{z}}(\mathcal{L}(\mathcal{V}_{n+1}))$ such that $X = (\psi_\xi)_* Y$. Then, since the forms $\tilde{\Omega}_l$ and $\mathrm{d}\varphi_l$ are invariant under translations, we have

$$-\mathrm{d}\varphi_{l|\bar{z}} = (\psi_\xi)^*_{\bar{z}}(-\mathrm{d}\varphi_{l|z}) = (\psi_\xi)^*_{\bar{z}}(X \rfloor \tilde{\Omega}_{l|z}) = Y \rfloor \tilde{\Omega}_{l|\bar{z}}$$

so that $Y \in T_{\bar{z}}(\mathcal{L}(\mathcal{V}_{n+1}))$ is an algebraic solution of equation (3.9) at $\bar{z}$, hence the result. □

Proposition 3.2. *Let $Y : M \to T(\mathcal{L}(\mathcal{V}_{n+1}))$ be a vector field on $\mathcal{L}(\mathcal{V}_{n+1})$ defined on the submanifold M and satisfying equation (3.9) at each $z \in M$. Then, there exists a vector field $X : M \to T(\mathcal{L}(\mathcal{V}_{n+1}))$ invariant under translations along the fibres (and thus π-projectable) still satisfying equation (3.9).*

Proof. Take any (global) section $\sigma : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ (not necessarily the Lagrangian one) and consider the intersection $\Sigma := \sigma(j_1(\mathcal{V}_{n+1})) \cap M$. Then, by proposition 3.1, M coincides with the totality of fibres $\pi^{-1}(\pi(z))$ $z \in \Sigma$.

Consider the restriction of the field Y to the points of Σ and 'move' it along the fibres by means of a push forward under the structural group. The result is a vector field X on M satisfying the stated requirement. □

Definition 3.1. *A pair (M, Y) satisfying the requirement of proposition 3.2 will be called an algebraic solution of equation (3.9). In a similar way, an algebraic solution of equations (2.22) will be understood as a vector field on $j_1(\mathcal{V}_{n+1})$, defined on a submanifold N of $j_1(\mathcal{V}_{n+1})$ and satisfying equations (2.22) everywhere on N.*

Proposition 3.3. *Equation (3.9) admits an algebraic solution if and only if equations (2.22) do.*

Proof. Let

$$M_0 := \{\, z \in \mathcal{L}(\mathcal{V}_{n+1}) \mid \dot{u}(z) = L(\pi(z)) \,\} \tag{3.11}$$

denote the image of $j_1(\mathcal{V}_{n+1})$ under the Lagrangian section l. Every algebraic solution $(M, \tilde{X})$ of equation (3.9) satisfies

$$0 = \langle \tilde{X}, \tilde{X} \rfloor \tilde{\Omega}_l \rangle = -\langle \tilde{X}, \mathrm{d}(\dot{u} - L) \rangle \tag{3.12}$$

everywhere on M, and, therefore, also on $\Sigma := M \cap M_0$. It follows that $\tilde{X}_{|\Sigma}$ is tangent to M_0. Accordingly, there exists a unique vector field $X : \pi(\Sigma) \to T(j_1(\mathcal{V}_{n+1}))$ π-related to $\tilde{X}$, i.e. satisfying $X = \pi_*(\tilde{X})$, or, which is the same, $l_*(X) = \tilde{X}_{|\Sigma}$. A straightforward argument, left to the reader, shows that X is then a solution of equations (2.22) on $\pi(\Sigma)$ $(= \pi(M))$.

Conversely, if (N, X) is an algebraic solution of equation (2.22) the horizontal lift X_l (3.3) satisfies equation (3.9) on the submanifold $\pi^{-1}(N) \subset \mathcal{L}(\mathcal{V}_{n+1})$. Indeed, the push forward $l_*(X)$ satisfies equation (3.9) on the image space $l(N) \subset \mathcal{L}(\mathcal{V}_{n+1})$, while the lift X_l is obtained by 'moving' $l_*(X)$ along the fibres by means of the structural group. The required conclusion is then a straightforward consequence of proposition 3.1. □

3.2. The constraint algorithm

In the previous subsection we have established the algebraic equivalence between equation (3.9) and equations (2.22) both in the regular and in the singular case.

Making use of some general results on presymplectic manifolds [1], we shall now extend to time-dependent degenerate Lagrangian systems the constraint algorithm proposed in [1–3].

The aim is to obtain, in the singular case, necessary and sufficient conditions for the solvability of equation (3.9) (and, consequently, of equations (2.22)) in a differential sense.

To start with, taking proposition 3.1 into account, we observe that there is no loss in generality in focusing attention on the surface M_0, i.e. in looking for solutions of equation (3.9) restricted to the points of M_0.

In fact, on one hand, the integral curves of any (kinematically admissible) solution X of equation (3.9) have to be π-related to the dynamical trajectories of the system in the velocity space $j_1(\mathcal{V}_{n+1})$; on the other hand, from equation (3.12), it is convenient keeping in mind that if $X \in T_{M_0}(\mathcal{L}(\mathcal{V}_{n+1}))$, then X is automatically tangent to M_0 and thus π-projectable onto $j_1(\mathcal{V}_{n+1})$.

In view of this, let us assume (a) the existence of points of M_0 on which equation (3.9) admits a solution X, and (b) that the totality of these points form a submanifold M_1 of M_0†.

Denoting by $\flat : T(\mathcal{L}(\mathcal{V}_{n+1})) \to T^*(\mathcal{L}(\mathcal{V}_{n+1}))$ the linear map $\flat(X) = X^\flat := X \,\lrcorner\, \tilde{\Omega}_l$, from equation (3.9) and equation (3.12) it is easily seen that the submanifold M_1 coincides with the subset

$$M_1 = \{\, z \in M_0 \mid \mathrm{d}\varphi_l(z) \in (T(M_0))^\flat \,\}. \tag{3.13}$$

Of course, as is well known [1–5], this merely algebraic characterization is not sufficient, in general, to ensure the existence of solutions $X : M_1 \to T(\mathcal{L}(\mathcal{V}_{n+1}))$ *tangent* to M_1, a requirement which is obviously necessary for a solution X to be dynamically significant. We are therefore forced to restrict our attention to the (generally smaller) submanifold

$$M_2 := \{\, z \in M_1 \mid \mathrm{d}\varphi_l(z) \in (T(M_1))^\flat \,\} \tag{3.14}$$

formed by the totality of points of M_1 at which the required tangency condition is satisfied. However, then, again, we have to require that at least one solution X should be tangent to M_2. By iterating the process, we generate a decreasing sequence of *constraint manifolds*

$$\mathcal{L}(\mathcal{V}_{n+1}) \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots \tag{3.15}$$

each embedded in the previous one and defined by the condition

$$M_k := \{\, z \in M_{k-1} \mid \mathrm{d}\varphi_l(z) \in (T(M_{k-1}))^\flat \,\} \qquad k \geqslant 1. \tag{3.16}$$

As pointed out in [1, 4, 5], three kinds of outcomes may then occur:

(a) there exists an integer $\bar{k} > 0$ such that $M_{\bar{k}} = \emptyset$. In this case equation (3.9) has no differential solution and the Lagrangian section l does not represent the dynamics of any systems;
(b) there exists an integer $\bar{k} > 0$ such that $M_{\bar{k}} \neq \emptyset$ but $\dim M_{\bar{k}} = 0$. This means that the submanifold $M_{\bar{k}}$ consists of isolated points, on which equation (3.9) admits $X = 0$ as the only possible solution;
(c) there exists an integer $\bar{k} > 0$ such that $M_{\bar{k}+1} = M_{\bar{k}}$ and $\dim M_{\bar{k}} > 0$. In this case, by construction, equation (3.9) admits at least one differential solution on $M_{\bar{k}}$, i.e. a solution tangent to $M_{\bar{k}}$ itself. Following [1–5], we shall set $M := M_{\bar{k}}$ and call M the *final constraint manifold*. Of course, the last case is the only dynamically interesting one.

For later use, it is worth pointing out an alternative characterization of the constraint submanifolds M_k, based on an algorithm different from the one employed in equation (3.16).

To this end, we recall the following [1].

Proposition 3.4. *Let $i : N \to M$ be a submanifold of a finite-dimensional presymplectic manifold (M, ω). Then, denoting by $TN^\perp := \{\, X \in T_N M \mid \omega(X, Y) = 0\ \forall\, Y \in TN \,\}$ the presymplectic complement of TN, one has the identifications*

- $TN^\perp = \{\, X \in T_N M \mid i^*(X \,\lrcorner\, \omega) = 0 \,\}$
- $(TN^\perp)^0 = (TN)^\flat$, *$(TN^\perp)^0$ denoting the annihilator of $TN^\perp$.*

† A similar assumption will be tacitly extended to every other subset $M_k \subset M_0$ arising in the course of the subsequent discussion.

Making use of proposition 3.4, it is a straightforward matter (left to the reader) to prove the relations†

$$M_{k+1} = \{ z \in M_k \mid \langle TM_k^{\perp}, \mathrm{d}\varphi_l \rangle(z) = 0 \}. \tag{3.17}$$

In connection with this, referring once again to [1] for proofs and related comments, we note that, whenever the algorithm (3.17) terminates with a final constraint submanifold M of dimension > 0, the relation

$$\langle TM^{\perp}, \mathrm{d}\varphi_l \rangle(z) = 0 \tag{3.18}$$

holds identically $\forall z \in M$. Moreover, it is also seen that such a submanifold M is automatically *maximal*, i.e. if N is any other submanifold of M_0 along which equation (3.9) possesses a differential solution, then N is contained in M.

The algorithm outlined above provides a constructive method for finding differential solutions for equation (3.9) or, equivalently, for equations (2.22) in the singular case.

Of course, when existing, such solutions are in general non-unique, but are determined up to vector fields belonging to $\ker \tilde{\Omega}_l \cap TM$.

Moreover, in the stated algorithm, nothing ensures that the solutions are kinematically admissible, i.e. that they do effectively represent dynamical flows (or semisprays, or SODEs). The search for SODE solutions, known as the 'second-order differential equation problem', will be dealt with in the next subsection.

3.3. The second-order differential equation problem

After analysing the solvability of the equations of motion in the differential sense, we shall now discuss under what circumstances they have a *dynamical* significance.

This requires examining under what conditions there exists a submanifold S of the final constraint submanifold M along which equation (3.9) admits kinematically admissible solutions.

The argument is dealt with in [3] for the autonomous case. Our plan is to extend it to the present, time-dependent, geometrical context.

To start with, given a (degenerate) Lagrangian section l, let us consider the involutive distribution‡ $D := \ker \Omega_l \cap V(j_1(\mathcal{V}_{n+1})) \subset T(j_1(\mathcal{V}_{n+1}))$, Ω_l denoting the Poincaré–Cartan 2-form (2.21) associated with l, and $V(j_1(\mathcal{V}_{n+1}))$ being the vertical bundle over $j_1(\mathcal{V}_{n+1})$. In local coordinates, from the expression

$$\Omega_l = \frac{\partial L}{\partial q^i}\, \mathrm{d}q^i \wedge \mathrm{d}t + \frac{\partial^2 L}{\partial t\, \partial \dot{q}^i}\, \mathrm{d}t \wedge \omega^i + \frac{\partial^2 L}{\partial q^j\, \partial \dot{q}^i}\, \mathrm{d}q^j \wedge \omega^i + \frac{\partial^2 L}{\partial \dot{q}^j\, \partial \dot{q}^i}\, \mathrm{d}\dot{q}^j \wedge \omega^i$$

it is easily seen that every vector V belonging to D is necessarily of the form

$$V = V^i \frac{\partial}{\partial \dot{q}^i} \tag{3.19a}$$

with the components V^i subject to the conditions

$$V^i \frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j} = 0 \qquad j = 1, \ldots, n. \tag{3.19b}$$

In connection with the distribution D we introduce the following:

† The proof is easily obtained by adapting to the present context the general formalism of presymplectic geometry developed in [1].

‡ Here it is systematically supposed that the rank of D is constant everywhere, i.e. in view of the subsequent equation (3.19*b*), that rank $\left\| \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right\| = r < n$ constant.

Definition 3.2. *A Lagrangian section $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ is called admissible if and only if the leaf space $\mathfrak{I} := j_1(\mathcal{V}_{n+1})/D$ of the foliation generated by D admits a manifold structure such that the canonical projection $\rho : j_1(\mathcal{V}_{n+1}) \to \mathfrak{I}$ is a submersion*†.

We next consider the following basic facts.

- The assignment of a Lagrangian section $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ determines a foliation of $\mathcal{L}(\mathcal{V}_{n+1})$ in terms of the one-parameter family of leaves

$$M_\xi := \{ z \in \mathcal{L}(\mathcal{V}_{n+1}) \mid \dot{u}(z) = L(\pi(z)) + \xi \} \qquad \xi \in \Re. \tag{3.20}$$

  Every such leaf is clearly the image of $j_1(\mathcal{V}_{n+1})$ under the section $l_\xi : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ described locally by $\dot{u} = L(t, q^i, \dot{q}^i) + \xi$.
- Let $D_l := \ker \tilde{\Omega}_l \cap V(\mathcal{L}(\mathcal{V}_{n+1}))$ be the involutive distribution in $\mathcal{L}(\mathcal{V}_{n+1})$, the intersection between the kernel of $\tilde{\Omega}_l$ and the vertical bundle over $\mathcal{L}(\mathcal{V}_{n+1})$. Taking the representation

$$\tilde{\Omega}_l = \mathrm{d}\dot{u} \wedge \mathrm{d}t + \frac{\partial^2 L}{\partial t \partial \dot{q}^i} \mathrm{d}t \wedge \omega^i + \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i} \mathrm{d}q^j \wedge \omega^i + \frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i} \mathrm{d}\dot{q}^j \wedge \omega^i - \frac{\partial L}{\partial \dot{q}^i} \mathrm{d}\dot{q}^i \wedge \mathrm{d}t$$

  into account, it is easily seen that every vector $V \in D_l$ is expressed locally as

$$V = V^i \left(\frac{\partial L}{\partial \dot{q}^i} \frac{\partial}{\partial \dot{u}} + \frac{\partial}{\partial \dot{q}^i} \right) \tag{3.21}$$

  the components V^i obeying the requirement (3.19*b*). This shows that, for every $z \in \mathcal{L}(\mathcal{V}_{n+1})$, the vector subspace $D_{l|z} \subset T_z(\mathcal{L}(\mathcal{V}_{n+1}))$ is nothing but the horizontal lift (3.3) of the vector subspace $D_{|\pi(z)} \subset T_{\pi(z)}(j_1(\mathcal{V}_{n+1}))$. By equation (3.21) it follows that the distribution D_l is automatically tangent to every surface (3.20), being $D_{lz} = l_{\xi *}(D_{|\pi(z)})$ for every $z \in M_\xi$. Thus the leaves of the foliation generated by D_l lie on the surfaces (3.20).

In particular, whenever the Lagrangian section l is admissible, a direct consequence of the previous analysis is that the quotient space $\mathfrak{L} := \mathcal{L}(\mathcal{V}_{n+1})/D_l$ has a manifold structure and that the canonical projection $\xi : \mathcal{L}(\mathcal{V}_{n+1}) \to \mathfrak{L}$ is a submersion.

More specifically, $\mathfrak{L}$ is a principal fibre bundle over $\mathfrak{I}$, with structural group isomorphic to $(\Re, +)$.

The situation is summarized into the commutative diagram

$$\begin{array}{ccc} \mathcal{L}(\mathcal{V}_{n+1}) & \xrightarrow{\pi} & j_1(\mathcal{V}_{n+1}) \\ {\scriptstyle \xi}\big\downarrow & & \big\downarrow{\scriptstyle \rho} \\ \mathfrak{L} & \xrightarrow{\pi} & \mathfrak{I} \end{array} \tag{3.22}$$

in which the horizontal arrows indicate principal fibrations, while the vertical ones denote canonical quotient maps.

We now observe that, in view of equation (3.21), in addition to $D_l \,\lrcorner\, \tilde{\Omega}_l = 0$, the distribution D_l also satisfies $\langle D_l, \mathrm{d}\varphi_l \rangle = 0$, $\varphi_l = \dot{u} - L$ denoting the trivialization of $\mathcal{L}(\mathcal{V}_{n+1})$ induced by the Lagrangian section. From this, following [7, 8], we conclude that there exists a presymplectic 2-form $\bar{\Omega}_l$ over $\mathfrak{L}$ and a trivialization $\bar{\varphi}_l$ of the principal fibre bundle $\pi : \mathfrak{L} \to \mathfrak{I}$, such that $\tilde{\Omega}_l = \xi^*(\bar{\Omega}_l)$ and $\varphi_l = \xi^*(\bar{\varphi}_l)$.

† Definition 3.2 generalizes that given in [3] concerning time-independent Lagrangians. In particular, as pointed out in [3], the condition of admissibility is weaker than the requirement of almost regularity employed in [2] and, for time-dependent systems, in [5].

We may therefore define a reduced problem of motion on the quotient space $\mathfrak{L}$, based on the equation

$$\bar{Z} \rfloor \bar{\Omega}_l = -\mathrm{d}\bar{\varphi}_l \tag{3.23}$$

for the unknown $\bar{Z} \in D^1(\mathfrak{L})$.

Concerning the solvability of equation (3.23), we may apply again the presymplectic constraint algorithm outlined in section 3.2†.

The connection between the dynamics on $\mathcal{L}(\mathcal{V}_{n+1})$ and on $\mathfrak{L}$ is then summarized into the following equivalence theorem.

Theorem 3.1.

(a) *The presymplectic algorithm terminates with a final constraint submanifold M in $\mathcal{L}(\mathcal{V}_{n+1})$ if and only if the corresponding 'reduced' algorithm terminates with a final constraint submanifold $\bar{M}$ in $\mathfrak{L}$.*

(b) *The problem of motion (3.9) is equivalent to the reduced one (3.23), in the sense that:*

1. *for every solution X of equation (3.9), if $\xi_*(X)$ exists, it satisfies the reduced equation (3.23);*
2. *if $\bar{X}$ satisfies equation (3.23), then every X ξ-related to $\bar{X}$ solves equation (3.9).*

The proof is essentially identical to that given in [2]. Strictly speaking, the analysis presented in [2] refers to the autonomous case, under the assumption of almost regularity of the Lagrangian. However, it is an easy matter to verify that the same arguments apply equally well to the present context.

By analogy with the terminology adopted in [3], we shall call *prolongable* every vector field X on $\mathcal{L}(\mathcal{V}_{n+1})$ which projects to $\mathfrak{L}$.

The previous theorem is then a statement on the existence of prolongable solutions of equation (3.9). Indeed, if $\bar{X}$ solves equation (3.23), then any vector field X ξ-related to $\bar{X}$ is a solution of equation (3.9), projecting to $\mathfrak{L}$.

A vector field X on $\mathcal{L}(\mathcal{V}_{n+1})$ will be said to be *semi-prolongable* if it is prolongable modulo $V(\mathcal{L}(\mathcal{V}_{n+1}))$.

Returning now to the surface $M_0 \subset \mathcal{L}(\mathcal{V}_{n+1})$, let $M \subset M_0$ be the final constraint submanifold associated with l through the algorithm discussed in section 3.2. We then have

Proposition 3.5. *The restriction $D_{l|M}$ is an involutive distribution in TM, foliating M. The corresponding leaf space $\mathfrak{M} := M/D_{l|M}$ is a submanifold embedded in $\mathfrak{L}$ and the induced projection $\xi_M : M \to \mathfrak{M}$ is a submersion.*

Proof. Proceeding as in the proof of proposition 1 in [3], we prove the inclusion $D_{l|M} \subseteq TM$ by induction on the constraint submanifolds M_k. To start the induction, we observe that $D_{l|M_0} \subseteq TM_0$, since the distribution D_l is automatically tangent to M_0. Let us now assume $D_{l|M_k} \subseteq TM_k$. In view of (3.17), the constraint submanifold M_{k+1} is characterized by the vanishing of functions of the form $\phi = \langle Z, \mathrm{d}\varphi_l\rangle$, with $Z \in TM_k^{\perp}$. Therefore, given any vector field Y belonging to D_l, Y is tangent to M_{k+1} iff $Y(\phi)_{|M_{k+1}} = 0$ for all such ϕ. However,

$$Y(\langle Z, \mathrm{d}\varphi_l\rangle) = \langle [Y, Z], \mathrm{d}\varphi_l\rangle + \langle Z, L_Y\, \mathrm{d}\varphi_l\rangle. \tag{3.24}$$

† For simplicity, we apply the algorithm to the whole presymplectic manifold $(\mathfrak{L}, \bar{\Omega}_l)$ [1].

Since D_l is tangent to the leaves (3.20), the second term in the right-hand side of equation (3.24) vanishes identically. Furthermore, if W denotes an arbitrary vector field belonging to TM_k, by the assumption on Y, Z and W, we have along M_k

$$\tilde{\Omega}_l([Y, Z], W) = -L_Y(\tilde{\Omega}_l)(Z, W) + L_Y(\tilde{\Omega}_l(Z, W)) - \tilde{\Omega}_l(Z, [Y, W]) = 0.$$

It follows that $[Y, Z]_{|M_k} \in TM_k^{\perp}$, whence, again by equation (3.17), $\langle [Y, Z], \mathrm{d}\varphi_l \rangle_{|M_{k+1}} = 0$. Comparison with equation (3.24) implies $Y(\langle Z, \mathrm{d}\varphi_l \rangle)_{|M_{k+1}} = 0$, showing that every $Y \in D_{l\,|M_{k+1}}$ is automatically tangent to M_{k+1}, i.e. that $D_{l\,|M_{k+1}} \subseteq TM_{k+1}$. By induction, this proves that $D_{|M}$ foliates M. Then, denoting by $i : M \to \mathcal{L}(\mathcal{V}_{n+1})$ the embedding, we can identify the quotient space $\mathfrak{M} = M/D_{l\,|M}$ with the image of M under the map $\xi_M := \xi \circ i$. From this it follows that $\mathfrak{M}$ inherits an embedded submanifold structure from $\mathfrak{L}$ and that ξ_M is a submersion. □

Proposition 3.6. *Let $X \in D^1(M)$ be a vector field satisfying equation (3.9). Then $\tilde{J}(X) \in D_{l\,|M}$*

Proof. By definition (see equation (3.8)), $\tilde{J}(X) = \langle X, \omega^i \rangle \big(\frac{\partial L}{\partial \dot{q}^i} \frac{\partial}{\partial u} + \frac{\partial}{\partial \dot{q}^i} \big)$. Moreover, by direct calculation, it is easily seen that every solution X of equation (3.9) satisfies the condition $\langle X, \omega^i \rangle \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = 0$ $j = 1, \ldots, n$. The result then follows from equations (3.19*b*) and (3.21). □

At this point, noticing that the relation

$$\tilde{J}(X) = 0 \tag{3.25}$$

characterizes every vector field $X \in D^1(M_0)$ π-related to a dynamical flow on $j_1(\mathcal{V}_{n+1})$, we state the following important proposition, solving in an 'ultra-pointwise' sense the existence problem for SODE solutions of equation (3.9)†.

Proposition 3.7. *Let l be an admissible Lagrangian section and $X \in D^1(M)$ a semi-prolongable solution of equation (3.9). Then there exists a unique point in each leaf of the foliation of M generated by $D_{l\,|M}$ at which X is a SODE.*

Proof. First of all, denoting by $X = \frac{\partial}{\partial t} + X^i \frac{\partial}{\partial q^i} + \dot{X}^i \frac{\partial}{\partial \dot{q}^i} + X^u \frac{\partial}{\partial u}$ a semi-prolongable solution of equation (3.9)‡, we note that, by definition, the components X^i are constant on the leaves of the foliation generated by $D_{l\,|M}$. With this in mind, let us consider the vertical vector field

$$-\tilde{J}(X) = -\langle X, \omega^i \rangle \left(\frac{\partial L}{\partial \dot{q}^i} \frac{\partial}{\partial u} + \frac{\partial}{\partial \dot{q}^i} \right).$$

In view of proposition 3.6, we have $-\tilde{J}(X) \in D_{l\,|M}$. The integral curves of $-\tilde{J}(X)$ are therefore vertical trajectories $\gamma(\sigma) : (t, q^i, \dot{q}^i(\sigma), \dot{u}(\sigma))$, contained in the leaves of the foliation, and determined locally by differential equations of the form

$$\frac{\mathrm{d}\dot{q}^i}{\mathrm{d}\sigma}(\sigma) = \dot{q}^i(\sigma) - X^i$$
$$\frac{\mathrm{d}\dot{u}}{\mathrm{d}\sigma}(\sigma) = \frac{\partial L}{\partial \dot{q}^i}(t, q^j, \dot{q}^j(\sigma))(\dot{q}^i(\sigma) - X^i).$$

† Once again, proposition 3.7 generalizes a result stated in [3].

‡ By equation (3.9), it is easily seen that the component of X along $\frac{\partial}{\partial t}$ is necessarily $X^0 = 1$.

Now, set $m = (t_m, q^i_m, \dot{q}^i_m, L(t_m, q^i_m, \dot{q}^i_m)) \in M$ and denote by $\mathfrak{L}_m \subset M$ the leaf through m. From the previous equations, it is straightforward to deduce that the integral curve $\gamma_m(\sigma)$ starting at m for $\sigma = 0$ is expressed as

$$\dot{q}^i(\sigma) = X^i + \exp(\sigma)(\dot{q}^i_m - X^i)$$

$$\dot{u}(\sigma) = L(t_m, q^i_m, \dot{q}^i(\sigma)).$$

As $\sigma \to -\infty$, $\dot{q}^i(\sigma) \to X^i$ and $\dot{u}(\sigma) \to L(t_m, q^i_m, X^i)$, so that $n_X := (t_m, q^i_m, X^i, L(t_m, q^i_m, X^i))$ is a limit point of the curve $\gamma_m(\sigma)$. Since $\gamma_m(\sigma) \in \mathfrak{L}_m\ \forall\sigma$, and $\mathfrak{L}_m$ is closed, we conclude that $n_X \in \mathfrak{L}_m$. Moreover, since $X^i =$ constant on $\mathfrak{L}_m$, n_X is independent of the choice of the point $m \in \mathfrak{L}_m$.

Finally, by construction, it is clear that n_X is the unique point in the leaf $\mathfrak{L}_m$ at which X is (π-related to) a second-order differential equation, i.e. $\tilde{J}(X)(n_X) = 0$. □

In what follows, two semi-prolongable solutions X, Y of equation (3.9) will be called $\tilde{J}$-equivalent if they satisfy the condition $\tilde{J}(X) = \tilde{J}(Y)$.

This defines an equivalence relation whose $\tilde{J}$-equivalence classes will be denoted by $[X]$. From the proof of proposition 3.7 it is easily seen that the point n_X depends on $[X]$ rather than on X. In other words, if $X, Y \in [X]$, then $n_X = n_Y$. According to this we shall use the notation $n_{[X]}$.

Now, let X be a semi-prolongable solution and let $S_{[X]}$ denote the union of all the points $n_{[X]}$, one for each leaf.

By proposition 3.7, there exists an injection $\alpha_{[X]} : \mathfrak{M} \to M$ given by $\alpha_{[X]}(\tilde{m}) = n_{[X]}(m)$ with $n_{[X]}(m)$ defined by starting from any $m \in \xi_M^{-1}(\tilde{m})$.

The image of $\mathfrak{M}$ under $\alpha_{[X]}$ is clearly identical to the set $S_{[X]}$. Moreover, in local coordinates, it is easily seen that $T\alpha_{[X]}$ is non-singular. The conclusion is that $S_{[X]}$ is a submanifold of M diffeomorphic to $\mathfrak{M}$. Note that if $[X] \neq [Y]$, then $S_{[X]} \neq S_{[Y]}$.

Therefore, for a fixed semi-prolongable solution X of equation (3.9), we have found a submanifold $S_{[X]}$ of M along which X satisfies both equations (3.9) and (3.25).

Unfortunately, this is not enough, because, in general, $X_{|S_{[X]}}$ does not need to be tangent to $S_{[X]}$. However, we observe that, by construction, one has the direct sum decomposition $T_{S_{[X]}}M = TS_{[X]} \oplus D_{l|S_{[X]}}$. Consequently, there is a unique decomposition of X of the form $X_{|S_{[X]}} = \bar{X} + V$ with $\bar{X} \in TS_{[X]}$ and $V \in D_{l|S_{[X]}}$. Since $V \in D$, it follows that $\bar{X}$ is a SODE solution of equation (3.9) along $S_{[X]}$.

The following last proposition concerns the uniqueness of kinematically admissible solutions on each submanifold $S_{[X]}$.

Proposition 3.8. *There exists a unique vector field Y tangent to $S_{[X]}$, simultaneously satisfying equations (3.9) and (3.25).*

Proof. Let Y and X denote two such vector fields. Then $0 = \tilde{J}(Y) = \tilde{J}(Z)$ which implies $Y - Z \in D_{l|S_{[X]}}$. However, then $Y = Z$ because $TS_{[X]} \cap D_{l|S_{[X]}} = \{0\}$. □

Finally, we may state the following theorem, summarizing the whole content of the previous discussion

Theorem 3.2. *Let $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ be an admissible (degenerate) Lagrangian section with final constraint submanifold M embedded in $M_0 \subset \mathcal{L}(\mathcal{V}_{n+1})$. Then there exists at least one* *submanifold S of M and a unique (for fixed S) vector field X tangent to S which simultaneously satisfies*

$$X \rfloor \tilde{\Omega}_l + \mathrm{d}\varphi_{l\,|S} = 0$$

$$\tilde{J}(X)_{|S} = 0.$$

Moreover, every such submanifold S is diffeomorphic to $\mathfrak{M}$.

3.4. Example

We conclude this paper with a simple example illustrating our procedure.

Consider a four-dimensional configuration spacetime $\mathcal{V}_{3+1} \simeq \Re^4$, referred to (global) coordinates t, x, y, z. Every choice of trivialization u of the associated bundle of affine scalars $P \to \mathcal{V}_{3+1}$ induces, respectively, on $j_1(\mathcal{V}_{3+1})$ and $\mathcal{L}(\mathcal{V}_{3+1})$ fibred coordinates of the form $t, x, y, z, \dot{x}, \dot{y}, \dot{z}$ and $t, x, y, z, \dot{x}, \dot{y}, \dot{z}, \dot{u}$.

Now, let $l : j_1(\mathcal{V}_{3+1}) \to \mathcal{L}(\mathcal{V}_{3+1})$ be the singular Lagrangian section having a local representation

$$\dot{u} = L(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) := \tfrac{1}{12}\dot{x}^4 - \dot{x}\dot{y} - \tfrac{1}{2}\dot{y}^2 + \dot{z} + z(\dot{y} + t) + x. \tag{3.26}$$

Taking equation (3.5) into account, the curvature 2-form of the connection generated by l is expressed (up to a sign) as

$$\begin{aligned}\tilde{\Omega}_l = {} & \mathrm{d}\dot{u} \wedge \mathrm{d}t + \mathrm{d}\left(\tfrac{1}{3}\dot{x}^3 - \dot{y}\right) \wedge (\mathrm{d}x - \dot{x}\,\mathrm{d}t) - \mathrm{d}\,(\dot{y} + \dot{x} - z) \wedge (\mathrm{d}y - \dot{y}\,\mathrm{d}t) \\ & - \left(\tfrac{1}{3}\dot{x}^3 - \dot{y}\right) \mathrm{d}\dot{x} \wedge \mathrm{d}t + (\dot{y} + \dot{x} - z)\ \mathrm{d}\dot{y} \wedge \mathrm{d}t - \mathrm{d}\dot{z} \wedge \mathrm{d}t.\end{aligned} \tag{3.27}$$

The reader may easily verify that the closed 2-form (3.27) is presymplectic.

Then, recalling equation (3.9), the resulting equations of motion on the Lagrangian bundle $\mathcal{L}(\mathcal{V}_{3+1})$ are given by

$$\tilde{Z} \rfloor \tilde{\Omega}_l = -\mathrm{d}\dot{u} + \left(\tfrac{1}{3}\dot{x}^3 - \dot{y}\right) \mathrm{d}\dot{x} - (\dot{y} + \dot{x} - z)\ \mathrm{d}\dot{y} + \mathrm{d}\dot{z} + (\dot{y} + t)\ \mathrm{d}z + z\,\mathrm{d}t + \mathrm{d}x \tag{3.28}$$

with unknown

$$\tilde{Z} = Z^0 \frac{\partial}{\partial t} + Z^x \frac{\partial}{\partial x} + Z^y \frac{\partial}{\partial y} + Z^z \frac{\partial}{\partial z} + \dot{Z}^x \frac{\partial}{\partial \dot{x}} + \dot{Z}^y \frac{\partial}{\partial \dot{y}} + \dot{Z}^z \frac{\partial}{\partial \dot{z}} + \dot{Z}^u \frac{\partial}{\partial \dot{u}}.$$

As pointed out in section 3.2, in order to solve the problem of motion (3.28), it is sufficient to focus attention on the submanifold

$$M_0 := \{\, z \in \mathcal{L}(\mathcal{V}_{3+1}) \mid \dot{u}(z) = L(\pi(z)) \,\}$$

the image of $j_1(\mathcal{V}_{3+1})$ under the section l.

In view of this, a direct computation shows that equation (3.28) admits solutions only on the submanifold $M_1 \subset M_0$ expressed in coordinates as

$$M_1 := \{\, z \in M_0 \mid \dot{y}(z) + t(z) = 0 \,\}.$$

The resulting family of (algebraic) solutions consists of the totality of vector fields of the form

$$\tilde{Z} = Z_{|M_1} + Z(L) \frac{\partial}{\partial \dot{u}}_{|M_1} \tag{3.29a}$$

with

$$Z = \frac{\partial}{\partial t} + \dot{x}\,\frac{\partial}{\partial x} + \dot{y}\,\frac{\partial}{\partial y} + Z^z\,\frac{\partial}{\partial z} + \frac{Z^z + 1}{\dot{x}^2 + 1}\,\frac{\partial}{\partial \dot{x}} + \frac{\dot{x}^2 Z^z - 1}{\dot{x}^2 + 1}\,\frac{\partial}{\partial \dot{y}} + \dot{Z}^z\,\frac{\partial}{\partial \dot{z}} \tag{3.29b}$$

where Z^z and $\dot{Z}^z$ are arbitrary differentiable functions.

At this point, we have to check whether the family (3.29) includes at least one solution tangent to M_1. Indeed, by imposing the tangency requirement $\tilde{Z}(\dot{y}+t)_{|M_1} = 0$, it is easily seen that equation (3.28) possesses along M_1 infinite differential solutions $\hat{Z}$ of the form (3.29*a*) with

$$Z = \frac{\partial}{\partial t} + \dot{x}\,\frac{\partial}{\partial x} + \dot{y}\,\frac{\partial}{\partial y} - \frac{\partial}{\partial z} - \frac{\partial}{\partial \dot{y}} + \dot{Z}^z\,\frac{\partial}{\partial \dot{z}}. \tag{3.30}$$

We come to the conclusion that the constraint algorithm stops at the first step, with the final constraint manifold $M := M_1$.

Unfortunately, none of the solutions just found is a SODE on the whole submanifold M. We are therefore forced to implement all of the discussion concerning the second-order differential equation problem.

To start with, it is a straightforward matter (left to the reader) to check that the Lagrangian section (3.26) is admissible and that the involutive distribution $D_l := \ker\tilde{\Omega}_l \cap V(\mathcal{L}(\mathcal{V}_{3+1}))$ is (locally) generated by the vector field $\frac{\partial}{\partial \dot{u}} + \frac{\partial}{\partial \dot{z}}$, i.e. $D_l = Span\left(\frac{\partial}{\partial \dot{u}} + \frac{\partial}{\partial \dot{z}}\right)$.

As proved in proposition 3.5, the restriction $D_{l\,|M}$ foliates automatically the submanifold M. In this case, the corresponding leaf space $\mathfrak{M} := M/D_{l\,|M}$ may be referred, in a natural way, to local coordinates $t, x, y, z, \dot{x}$.

In view of this, it is easily seen that every solution $\hat{Z}$ is semi-prolongable and that all vector fields $\hat{Z}$ are $\tilde{J}$-equivalent. We have in fact†

$$-\tilde{J}(\hat{Z}) = (1+\dot{z})\left(\frac{\partial}{\partial \dot{u}} + \frac{\partial}{\partial \dot{z}}\right) \qquad \forall\, \hat{Z}. \tag{3.31}$$

Then, by starting from equation (3.31) and following the procedure stated in proposition 3.7, we can associate with the class $[\hat{Z}]$ of solutions (3.30) a unique submanifold $S_{[\hat{Z}]}$ on which all vector fields $\hat{Z}$ are (π-related to) SODEs. More precisely, $S_{[\hat{Z}]}$ is the image of the injection $\alpha_{[\hat{Z}]} : \mathfrak{M} \to M$ described in local coordinates by $\tilde{m} = (t, x, y, z, \dot{x}) \in \mathfrak{M} \to \alpha_{[\hat{Z}]}(\tilde{m}) := n_{[\hat{Z}]}(\tilde{m}) = (t, x, y, z, \dot{x}, \dot{y} = -t, \dot{z} = -1, \dot{u} = L(t, x, y, z, \dot{x}, -t, -1)) \in S_{[\hat{Z}]} \subset M$.

In general, the solutions $\hat{Z}$ will not be tangent to $S_{[\hat{Z}]}$. The final step consists then in considering the direct sum decomposition $T_{S_{[\hat{Z}]}}M = TS_{[\hat{Z}]} \oplus D_{l\,|S_{[\hat{Z}]}}$ and observing that every vector field $\hat{Z}$ admits a unique representation of the form $\hat{Z}_{|S_{[\hat{Z}]}} = \bar{Z}_{|S_{[\hat{Z}]}} + V_{|S_{[\hat{Z}]}}$ where

$$\bar{Z}_{|S_{[\hat{Z}]}} = \left(\frac{\partial}{\partial t} + \dot{x}\,\frac{\partial}{\partial x} + \dot{y}\,\frac{\partial}{\partial y} - \frac{\partial}{\partial z} - \frac{\partial}{\partial \dot{y}} + (-t + 2\dot{x})\,\frac{\partial}{\partial \dot{u}}\right)_{|S_{[\hat{Z}]}} \in TS_{[\hat{Z}]} \tag{3.32}$$

and

$$V_{|S_{[\hat{Z}]}} = \dot{Z}^z\left(\frac{\partial}{\partial \dot{u}} + \frac{\partial}{\partial \dot{z}}\right)_{|S_{[\hat{Z}]}} \in D_{l\,|S_{[\hat{Z}]}}.$$

The vector field (3.32) represents the unique kinematically admissible solution of equation (3.28) along the submanifold $S_{[\hat{Z}]}$.

† $\tilde{J} = \left(\left(\frac{1}{3}\dot{x}^3 - \dot{y}\right)\frac{\partial}{\partial \dot{u}} + \frac{\partial}{\partial \dot{x}}\right) \otimes (\mathrm{d}x - \dot{x}\,\mathrm{d}t) + \left(-(\dot{y} + \dot{x} - z)\,\frac{\partial}{\partial \dot{u}} + \frac{\partial}{\partial \dot{y}}\right) \otimes (\mathrm{d}y - \dot{y}\,\mathrm{d}y) + \left(\frac{\partial}{\partial \dot{u}} + \frac{\partial}{\partial \dot{z}}\right) \otimes (\mathrm{d}z - \dot{z}\,\mathrm{d}t)$.

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